Three-dimensional conformally flat homogeneous Lorentzian manifolds

2007 J. Phys. A: Math. Theor. 40 831

(http://iopscience.iop.org/1751-8121/40/4/017)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 831–851 doi:10.1088/1751-8113/40/4/017

Three-dimensional conformally flat homogeneous Lorentzian manifolds

Kyoko Honda[1] **and Kazumi Tsukada**[2]

[1] Graduate School of Humanities and Sciences, Ochanomizu University, 2-1-1 Otsuka Bunkyo-ku, Tokyo, 112-8610, Japan
[2] Department of Mathematics, Ochanomizu University, 2-1-1 Otsuka Bunkyo-ku, Tokyo, 112-8610, Japan

E-mail: honda@cc.ocha.ac.jp and tsukada@math.ocha.ac.jp

Received 13 October 2006
Published 9 January 2007
Online at stacks.iop.org/JPhysA/40/831

Abstract
We classify three-dimensional conformally flat homogeneous Lorentzian manifolds. Our classification depends on the form of the Ricci operators.

PACS numbers: 02.40.Ma, 02.40.Hw
Mathematics Subject Classification: 53C50, 53C30, 53A30, 53A15

1. Introduction

We are interested in the classification problem of conformally flat homogeneous semi-Riemannian manifolds. Takagi [7] classified the Riemannian case. That is, an n-dimensional simply connected conformally flat homogeneous Riemannian manifold is isometric to one of the following: (1) $M^n(k)$, (2) $M^m(k) \times M^{n-m}(-k)$, $k \neq 0$, $2 \leqslant m \leqslant n-2$, (3) $M^{n-1}(k) \times \mathbb{R}$, $k \neq 0$, where $M^m(k)$ denotes the simply connected complete Riemannian manifold of constant curvature k. Consequently, they are all symmetric spaces. In the previous paper [4], we studied conformally flat semi-Riemannian manifolds with $Q^2 = 0$, where Q denotes the Ricci operator, and showed how to construct such ones. We recall the method. We define an inner product $\langle\ ,\ \rangle$ of index $q+1$ on $\mathbb{R}^{n+2}$ by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i=1}^{k+1} \{x_i y_{k+1+i} + x_{k+1+i} y_i\} + \sum_{j=2(k+1)+1}^{n+2} \varepsilon_j x_j y_j,$$

where k is the fixed integer such that $1 \leqslant k \leqslant [n/2]$, $k \leqslant q$, and

$$\varepsilon_j = \begin{cases} -1 & 2(k+1)+1 \leqslant j \leqslant 2(k+1)+q-k \\ 1 & 2(k+1)+q-k+1 \leqslant k \leqslant n+2 \end{cases}$$

and denote by $\mathbb{R}^{n+2}_{q+1}$ an $(n+2)$-dimensional vector space endowed with this inner product $\langle\ ,\ \rangle$. We define the light cone Λ of $\left(\mathbb{R}^{n+2}_{q+1}, \langle\ ,\ \rangle\right)$ by

$$\Lambda = \{\boldsymbol{x} \in \mathbb{R}^{n+2} - \{0\} | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\}.$$

Let $\pi : \mathbb{R}^{n+2}_{q+1} \longrightarrow \mathbb{R}^{k+1}$ be the projection defined by $\pi(x_1, \ldots, x_{k+1}, x_{k+2}, \ldots, x_{n+2}) = (x_1, \ldots, x_{k+1})$. We denote by $\bar{\pi}$ the restriction of π to $\Lambda \cap \pi^{-1}(\mathbb{R}^{k+1} - \{0\})$. Then $\bar{\pi}$ is the fibre bundle over $\mathbb{R}^{k+1} - \{0\}$. Let N be a k-dimensional manifold and $F : N \to \mathbb{R}^{k+1} - \{0\}$ be a centro-affine hypersurface immersion. We consider the pull-back bundle of $\bar{\pi} : \Lambda \cap \pi^{-1}(\mathbb{R}^{k+1} - \{0\}) \to \mathbb{R}^{k+1} - \{0\}$ by the immersion F. We denote by M and f, the total space of the pull-back bundle and the bundle homomorphism of M into $\Lambda \cap \pi^{-1}(\mathbb{R}^{k+1} - \{0\})$, respectively. Then f is a hypersurface immersion of M into the light cone Λ. We proved that the induced metric on M by f is non-degenerate and that the semi-Riemannian manifold M with this metric is conformally flat and its Ricci operator Q satisfies $Q^2 = 0$ ([4] theorem 1.1). There are interesting relations between the semi-Riemannian geometry of M and the affine differential geometry of N. In particular if N is a homogeneous centro-affine hypersurface, then M is a homogeneous semi-Riemannian manifold ([4] theorem 2.1(3)). Thus unlike the Riemannian case we expect that there are various conformally flat homogeneous semi-Riemannian manifolds and think that it is not so easy to classify them. In this paper, we focus on the three-dimensional case and obtain the following result.

Theorem 1.1. *A three-dimensional simply connected conformally flat homogeneous Lorentzian manifold M^3_1 is isometric to one of the following six kinds of manifolds:*

(1) $M^3_1(k), k \in \mathbb{R}$,
(2) $M^2_1(k) \times \mathbb{R}^1, k \neq 0$,
(3) $\mathbb{R}^1_1 \times M^2(k), k \neq 0$,
where $M^m_1(k)$ is an m-dimensional simply connected homogeneous Lorentzian manifold of constant sectional curvature k and $M^m(k)$ is an m-dimensional simply connected homogeneous Riemannian manifold of constant sectional curvature k and $\mathbb{R}^1$(resp. $\mathbb{R}^1_1$) denote a one-dimensional real vector space with a positive (resp. negative) inner product.
(4) The universal covering $\widetilde{SL(2, \mathbb{R})}$ of $SL(2, \mathbb{R})$ with a left invariant Lorentzian metric. The bracket operation $[\,,]$ with respect to a semi-orthonormal basis $\{e_1, e_2, e_3\}$(all zero except $\langle e_1, e_2\rangle = \langle e_3, e_3\rangle = 1$) is given by

$$[e_1, e_2] = -ke_3$$
$$[e_2, e_3] = -\tfrac{\sqrt{3}}{2}ke_1 + \tfrac{1}{2}ke_2 \qquad k \neq 0$$
$$[e_3, e_1] = \tfrac{1}{2}ke_1 + \tfrac{\sqrt{3}}{2}ke_2.$$

(5) The nonunimodular Lie group with a left invariant Lorentzian metric. The bracket operation $[\,,]$ with respect to a semi-orthonormal basis $\{e_1, e_2, e_3\}$ is given by

$$[e_1, e_2] = 0$$
$$[e_2, e_3] = \frac{\varepsilon}{2k}e_1 - ke_2 \qquad k \neq 0 \quad \varepsilon = 1 \text{ or } -1$$
$$[e_3, e_1] = 3ke_1.$$

(6) The universal covering of $\Lambda \cap \pi^{-1}(c)$, where Λ is the light cone in $\mathbb{R}^5_2$ and $\pi : \mathbb{R}^5_2 \to \mathbb{R}^2$ is the projection and c is one of the following homogeneous centro-affine plane curves of $\mathbb{R}^2$:

1. $y = x^\lambda (\lambda > 1, x > 0)$,
2. $y = x^\lambda (\lambda \leqslant -1, x > 0)$,
3. $\begin{cases} x = e^t \cos bt \\ y = e^t \sin bt \end{cases} \quad (b > 0)$,
4. $x^2 + y^2 = 1$,
5. $y = x \log x \ (x > 0)$,

(see the construction described before this theorem).

The six classes in theorem 1.1 are characterized by their Ricci operators.

Corollary 1.2. *The Lorentzian manifolds in theorem 1.1 have the following form of their Ricci operators according to the number of the theorem:*

$$(1)\ \begin{pmatrix} 2k & & \\ & 2k & \\ & & 2k \end{pmatrix} k \in \mathbb{R} \qquad (2)\ \begin{pmatrix} k & & \\ & k & \\ & & 0 \end{pmatrix} k \neq 0$$

$$(3)\ \begin{pmatrix} 0 & & \\ & k & \\ & & k \end{pmatrix} k \neq 0 \qquad (4)\ \begin{pmatrix} k^2 & \sqrt{3}k^2 & \\ -\sqrt{3}k^2 & k^2 & \\ & & -2k^2 \end{pmatrix} k \neq 0$$

$$(5)\ \begin{pmatrix} -8k^2 & \varepsilon & \\ & -8k^2 & \\ & & -8k^2 \end{pmatrix} k \neq 0 \qquad (6)\ \begin{pmatrix} 0 & \varepsilon & \\ & 0 & \\ & & 0 \end{pmatrix} \varepsilon = 1 \text{ or } -1,$$

where the matrices of (1), (2) and (3) are those with respect to the orthonormal bases and the matrices of (4), (5) and (6) are those with respect to the semi-orthonormal bases.

Remark 1.3. Chaichi, García-Río and Vázquez-Abal [1] studied curvature properties of three-dimensional Lorentzian manifolds admitting a parallel degenerate line field. In particular they characterized those manifolds which are conformally flat. In our classification, the case (6) in theorem 1.1 consists of homogeneous Lorentzian manifolds whose image $\operatorname{Im} Q$ of the Ricci operator Q is a parallel degenerate line field. So our results show nice examples of such properties studied in [1].

This paper is organized as follows: in section 2 we show the identity of the eigenvalues of the tensor field $A = 1/(n-2)\{Q - S/(2(n-1))\,\mathrm{Id}\}$ on an n-dimensional conformally flat homogeneous semi-Riemannian manifold M (theorem 2.1), where Q and S denote the Ricci operator and the scalar curvature of M, respectively. Applying this identity, we give a local classification of conformally flat homogeneous semi-Riemannian manifolds with real diagonalizable Ricci operators (theorem 2.3) and a complete classification of possible candidates for the linear operators A of conformally flat homogeneous Lorentzian manifolds (theorem 2.4). The other sections are devoted to the proof of theorem 1.1. Our proof depends on the classification of the Ricci operators. The Ricci operator $Q_p (p \in M)$ of a three-dimensional Lorentzian manifold M_1^3 is known to have exactly one of the following four types (cf O'Neill [6] pp 261–262):

$$\text{case 1} \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}, \qquad \text{case 2} \begin{pmatrix} a & -b & \\ b & a & \\ & & \lambda \end{pmatrix} b \neq 0,$$
$$\text{case 3} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 1 & 0 & \lambda \end{pmatrix}, \qquad \text{case 4} \begin{pmatrix} \lambda & \varepsilon & \\ & \lambda & \\ & & \sigma \end{pmatrix} \varepsilon = 1 \text{ or } -1, \tag{1.1}$$

where the matrix of case 1 is the one with respect to an orthonormal basis and the matrices of cases 2–4 are those with respect to semi-orthonormal bases. Evidently, the Ricci operator of a three-dimensional homogeneous Lorentzian manifold has the same form at every point. In section 3, we deal with case 2 and show that it is isometric to the Lorentzian manifold of (4) in theorem 1.1. In section 4, we deal with case 3 and show that this case does not occur. In section 5, we study case 4 and show that $\lambda = \sigma \leqslant 0$. In the case of $\lambda < 0$, it is isometric to the Lorentzian manifold of (5) in theorem 1.1. Moreover we give a local description of the case $\lambda = 0$. In section 6, we give a global classification of the case 4 with $\lambda = \sigma = 0$ and obtain (6) in theorem 1.1. Finally in section 7, we give a global description of case 1 with $\lambda_1 = \lambda_2 \neq 0, \lambda_3 = 0$ or $\lambda_1 = 0, \lambda_2 = \lambda_3 \neq 0$ and show that they are isometric to (2) or (3) in theorem 1.1.

2. Conformally flat homogeneous semi-Riemannian manifolds

Let M_q^n be an $n(\geqslant 3)$-dimensional semi-Riemannian manifold of index q. We denote by ∇ the Levi-Civita connection of M and by R, Q and S the curvature tensor, the Ricci operator and the scalar curvature of M, respectively. We define the Weyl conformal curvature tensor field C of type (1,3) and c of type (1,2) of M by

$$C(X, Y)Z = R(X, Y)Z - \frac{1}{n-2}(QX \wedge Y + X \wedge QY)Z + \frac{S}{(n-1)(n-2)}(X \wedge Y)Z \quad (2.1)$$

$$c(X, Y) = (\nabla_X Q)Y - (\nabla_Y Q)X - \frac{1}{2(n-1)}(X(S)Y - Y(S)X), \quad (2.2)$$

where $X \wedge Y$ denotes the endomorphism defined by $(X \wedge Y)(Z) = \langle Y, Z\rangle X - \langle X, Z\rangle Y$. The following are well known:

- M_q^n is conformally flat if and only if C vanishes for $n \geqslant 4$.
- $C \equiv 0$ implies $c \equiv 0$ for $n \geqslant 4$.
- The tensor C vanishes identically for any three-dimensional semi-Riemanian manifold.
- M_q^3 is conformally flat if and only if $c \equiv 0$.

Now we assume that M_q^n is conformally flat. For convenience, we define a tensor field A of type (1,1) by

$$A = \frac{1}{n-2}\left\{Q - \frac{S}{2(n-1)}\,\mathrm{Id}\right\}, \quad (2.3)$$

where Id denotes the identity transformation. Then A is a symmetric linear endomorphism of the tangent space T_pM. Since M_q^n is conformally flat, by (2.1) and (2.2) we have

$$R(X, Y) = AX \wedge Y + X \wedge AY, \quad (2.4)$$

$$(\nabla_X A)Y = (\nabla_Y A)X. \quad (2.5)$$

From now on we assume that M_q^n is a homogeneous semi-Riemannian manifold. Then evidently, the—possibly complex—eigenvalues of A and their algebraic multiplicities are constant on M. It is a similar situation to the shape operators for isoparametric hypersurfaces in the semi-Riemannian space form. Hahn obtained the basic identity concerning principal curvatures of an isoparametric hypersurface ([3] theorem 2.9). We have the same result for the eigenvalues of A.

Theorem 2.1. *Let M_q^n be a conformally flat homogeneous semi-Riemannian manifold and $\lambda_1, \ldots, \lambda_r$ be the distinct eigenvalues of the tensor field A on M with algebraic multiplicities $m_1, \ldots, m_r$, respectively. If for $i \in \{1, \ldots, r\}$, the eigenvalue λ_i is real and the dimension of its eigenspace coincides with its algebraic multiplicity, then we have*

$$\sum_{j \neq i} m_j \frac{\lambda_j + \lambda_i}{\lambda_j - \lambda_i} = 0. \quad (2.6)$$

Proof. Our proof is inspired by Hahn [3]. Let λ be a real eigenvalue of the tensor field A. We assume that the dimension of its eigenspace coincides with its algebraic multiplicity m. For each point $p \in M$, we define $T_\lambda(p) = \mathrm{Ker}(A - \lambda \mathrm{Id})$. Then the semi-Riemannian metric restricted to $T_\lambda(p)$ is non-degenerate and we have the orthogonal decomposition $T_pM = T_\lambda(p) + T_\lambda^\perp(p)$. By (2.5), we see that the eigenspace distribution T_λ is completely integrable and that its leaves are totally geodesic semi-Riemannian submanifolds in M. The orthogonal complement $T_\lambda^\perp$ is parallel with respect to the Levi-Civita connection ∇ along the leaves of T_λ. For this totally geodesic foliation, we define the conullity operator C as a smooth section of $\mathrm{Hom}\big(T_\lambda, \mathrm{End}(T_\lambda^\perp)\big)$ (cf Ferus [2]). We denote by $\pi : TM \to T_\lambda^\perp$ the orthogonal projection. Define a linear homomorphism C of $T_\lambda(p)$ into $\mathrm{End}\big(T_\lambda^\perp(p)\big)$ by

$$C_u x = -\pi(\nabla_x U) \qquad \text{for} \quad x \in T_\lambda^\perp(p), \quad u \in T_\lambda(p),$$

where U is a local smooth section of T_λ with $U_p = u$. At each point $p \in M$, we restrict a linear endomorphism $A - \lambda\,\mathrm{Id}$ to $T_\lambda^\perp(p)$ and denote it by Φ_λ. Then Φ_λ is a linear isomorphism of $T_\lambda^\perp(p)$. We have the following identity at each point $p \in M$:

$$\nabla_u \Phi_\lambda = \Phi_\lambda C_u \qquad \text{for} \quad u \in T_\lambda(p). \tag{2.7}$$

In fact for a local smooth section U of T_λ with $U_p = u$ and local smooth sections X and Y of $T_\lambda^\perp$ around p, we have

$$\begin{aligned}\langle (\nabla_U \Phi_\lambda)X, Y\rangle &= \langle (\nabla_U A)X, Y\rangle = \langle (\nabla_X A)U, Y\rangle \\ &= \langle (\lambda\,\mathrm{Id} - A)\nabla_X U, Y\rangle = \langle (\lambda\,\mathrm{Id} - A)\pi(\nabla_X U), Y\rangle \\ &= \langle (A - \lambda\,\mathrm{Id})C_u(X), Y\rangle = \langle \Phi_\lambda C_u(X), Y\rangle.\end{aligned}$$

Since the leaves of T_λ are totally geodesic, we have $R(x, v)v \in T_\lambda^\perp(p)$ for $x \in T_\lambda^\perp(p)$ and $v \in T_\lambda(p)$. For $v \in T_\lambda(p)$, we denote by $\widetilde{R}_v$ the linear endomorphism of $T_\lambda^\perp(p)$ defined by $x \mapsto R(x, v)v$ for $x \in T_\lambda^\perp(p)$. Let γ be a geodesic in a leaf of T_λ. Then it is known that the following identity holds (cf Ferus [2]):

$$\nabla_{\dot\gamma} C_{\dot\gamma} = C_{\dot\gamma}^2 + \widetilde{R}_{\dot\gamma}. \tag{2.8}$$

Moreover, we have the following identity.

Lemma 2.2. *For any $v \in T_\lambda(p)$,*

$$\mathrm{tr}\big(\widetilde{R}_v \Phi_\lambda^{-1}\big) = 0.$$

Proof of lemma 2.2. We take a geodesic $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p$ and $\dot\gamma(0) = v$. Then γ is a curve in the leaf of T_λ through p. Let $\{e_1, \ldots, e_{n-m}\}$ be a basis of $T_\lambda^\perp(p)$ and $\{E_1, \ldots, E_{n-m}\}$ be parallel frame fields of $T_\lambda^\perp$ along γ such that $(E_i)_p = e_i (i = 1, \ldots, n-m)$. We express Φ_λ, Φ_λ^{-1}, $C_{\dot\gamma}$ and $\widetilde{R}_{\dot\gamma}$ as $(n-m) \times (n-m)$-matrices with respect to $\{E_1, \ldots, E_{n-m}\}$ and denote them by the same notations. Differentiating Φ_λ and $C_{\dot\gamma}$ along γ, by (2.7) and (2.8) we have

$$\Phi_\lambda' = \Phi_\lambda C_{\dot\gamma}, \tag*{(2.7)$'$}$$

$$C_{\dot\gamma}' = C_{\dot\gamma}^2 + \widetilde{R}_{\dot\gamma}. \tag*{(2.8)$'$}$$

By these equations, we have

$$\big(\Phi_\lambda^{-1}\big)' = -\Phi_\lambda^{-1}\Phi_\lambda'\Phi_\lambda^{-1} = -\Phi_\lambda^{-1}(\Phi_\lambda C_{\dot\gamma})\Phi_\lambda^{-1} = -C_{\dot\gamma}\Phi_\lambda^{-1},$$

and

$$\begin{aligned}\big(\Phi_\lambda^{-1}\big)'' &= -C_{\dot\gamma}'\Phi_\lambda^{-1} - C_{\dot\gamma}\big(\Phi_\lambda^{-1}\big)' \\ &= -\big(C_{\dot\gamma}^2 + \widetilde{R}_{\dot\gamma}\big)\Phi_\lambda^{-1} - C_{\dot\gamma}\big(-C_{\dot\gamma}\Phi_\lambda^{-1}\big) = -\widetilde{R}_{\dot\gamma}\Phi_\lambda^{-1}.\end{aligned}$$

Thus we obtain the following:

$$\left(\Phi_\lambda^{-1}\right)'' = -\widetilde{R}_{\dot\gamma}\Phi_\lambda^{-1}. \tag{2.9}$$

Since M is a homogeneous semi-Riemannian manifold, the trace $\mathrm{tr}\left(\Phi_\lambda^{-1}\right)$ of the linear isomorphism Φ_λ^{-1} is constant on M. Therefore, we have $\mathrm{tr}\left(\left(\Phi_\lambda^{-1}\right)''\right) = \left(\mathrm{tr}\left(\Phi_\lambda^{-1}\right)\right)'' = 0$. By (2.9) it follows that $\mathrm{tr}\left(\widetilde{R}_{\dot\gamma}\Phi_\lambda^{-1}\right) = 0$. □

We continue the proof of theorem 2.1 We take a vector $v \in T_\lambda(p)$ such that $\langle v, v\rangle \neq 0$. Then by (2.4), $\widetilde{R}_v(x) = \langle v, v\rangle(A + \lambda\,\mathrm{Id})(x)$ for $x \in T_\lambda^\perp(p)$. By lemma 2.2, the trace of the linear endomorphism $(A + \lambda\,\mathrm{Id})(A - \lambda\,\mathrm{Id})^{-1}$ of $T_\lambda^\perp(p)$ is 0. From this, immediately we obtain the identity (2.6). □

As an application of theorem 2.1, we give a local classification of conformally flat homogeneous semi-Riemannian manifolds with real diagonalizable Ricci operators. This classification is same as that of the Riemannian case shown by Takagi [7].

Theorem 2.3. *Let M_q^n be an $n(\geqslant 3)$-dimensional conformally flat homogeneous semi-Riemannian manifold of index q whose Ricci operator is diagonalizable with respect to an orthonormal basis. Then M_q^n is isometric to one of the following:*

(1) A semi-Riemannian manifold of constant curvature.
(2) A semi-Riemannian manifold which is locally a product manifold of an m-dimensional semi-Riemannian manifold of constant curvature $k(\neq 0)$ and an $(n-m)$-dimensional semi-Riemannian manifold of constant curvature $-k$, where $2 \leqslant m \leqslant n-2$.
(3) A semi-Riemannian manifold which is locally a product manifold of an $(n-1)$-dimensional semi-Riemannian manifold of index $q-1$ of constant curvature $k(\neq 0)$ and a one-dimensional Lorentzian manifold or a product of an $(n-1)$-dimensional semi-Riemannian manifold of index q of constant curvature $k(\neq 0)$ and a one-dimensional Riemannian manifold.

Proof. The proof is quite similar to that of theorem A in Takagi [7]. So we show only its outline. Since the Ricci operator Q is real diagonalizable, the linear operator A is so. Therefore the eigenvalues $\lambda_1, \ldots, \lambda_r$ of A are all real and the dimensions of their eigenspaces coincide with their algebraic multiplicities. Hence we can apply the identity (2.6) for each $i \in \{1, \ldots, r\}$. Then we have

$$0 = \sum_{j\neq i} m_j \frac{\lambda_j + \lambda_i}{\lambda_j - \lambda_i} = \sum_{j\neq i} m_j \frac{\lambda_j^2 - \lambda_i^2}{(\lambda_j - \lambda_i)^2} \qquad \text{for each } i. \tag{2.10}$$

This implies that the linear operator A has at most two distinct eigenvalues. If A has only one eigenvalue λ, by (2.4) M is of constant curvature 2λ. If A has exactly two eigenvalues, by (2.10) the eigenvalues are λ and $-\lambda(\lambda > 0)$ and we have the orthogonal decomposition:

$$TM = T_\lambda + T_{-\lambda}$$

into the two eigenspace distributions T_λ and $T_{-\lambda}$. By the proof of theorem 2.1, T_λ and $T_{-\lambda}$ are parallel on M with respect to the Levi-Civita connection ∇. By the de Rham decomposition theorem, M_q^n is locally a product of two semi-Riemannian manifolds which are integral submanifolds of T_λ and $T_{-\lambda}$, respectively. □

As a second application, we give a classification of possible candidates for the linear operator A of conformally flat homogeneous Lorentzian manifolds (it is equivalent to the classification of the Ricci operators). The symmetric linear operator A in a Lorentzian vector space has exactly one of the following four forms (cf O'Neill [6] pp 261–262):

$$
\begin{aligned}
&(1)\quad \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} && \text{relative to an orthonormal basis} \\
&(2)\quad \begin{pmatrix} a & -b & & & \\ b & a & & & \\ & & \lambda_2 & & \\ & & & \ddots & \\ & & & & \lambda_{n-1} \end{pmatrix} && \begin{array}{l} b \neq 0 \\ \text{relative to an orthonormal basis} \\ \langle e_1, e_1\rangle = -1, \langle e_i, e_j\rangle = \delta_{ij} (i, j \geqslant 2) \end{array} \\
&(3)\quad \begin{pmatrix} \lambda_1 & 0 & 0 & & & \\ 0 & \lambda_1 & 1 & & & \\ 1 & 0 & \lambda_1 & & & \\ & & & \lambda_2 & & \\ & & & & \ddots & \\ & & & & & \lambda_{n-2} \end{pmatrix} && \begin{array}{l} \text{relative to a semi-orthonormal basis} \\ \langle e_1, e_2\rangle = 1, \langle e_i, e_j\rangle = \delta_{ij} (i, j \geqslant 3) \end{array} \\
&(4)\quad \begin{pmatrix} \lambda_1 & \varepsilon & & & \\ 0 & \lambda_1 & & & \\ & & \lambda_2 & & \\ & & & \ddots & \\ & & & & \lambda_{n-1} \end{pmatrix} && \begin{array}{l} \varepsilon = 1 \text{ or } -1 \\ \text{relative to a semi-orthonormal basis} \\ \langle e_1, e_2\rangle = 1, \langle e_i, e_j\rangle = \delta_{ij} (i, j \geqslant 3). \end{array}
\end{aligned}
\tag{2.11}
$$

Theorem 2.4. *Let M_1^n be an $n(\geqslant 3)$-dimensional conformally flat homogeneous Lorentzian manifold. Then the linear operator A defined by (2.3) has exactly one of the following four forms corresponding to (2.11):*

$$
(1)\quad \begin{pmatrix} \lambda & & & & & \\ & \ddots & & & & \\ & & \lambda & & & \\ & & & -\lambda & & \\ & & & & \ddots & \\ & & & & & -\lambda \end{pmatrix}
$$

$$
(2)\quad \begin{pmatrix} a & -b & & & & & & \\ b & a & & & & & & \\ & & \lambda & & & & & \\ & & & \ddots & & & & \\ & & & & \lambda & & & \\ & & & & & -\lambda & & \\ & & & & & & \ddots & \\ & & & & & & & -\lambda \end{pmatrix} \qquad \begin{array}{l} b \neq 0 \\ a^2 + b^2 = \lambda^2 \end{array}
$$

(3)
$$\begin{pmatrix} \lambda & 0 & 0 & & & & & & \\ 0 & \lambda & 1 & & & & & & \\ 1 & 0 & \lambda & & & & & & \\ & & & \lambda & & & & & \\ & & & & \ddots & & & & \\ & & & & & \lambda & & & \\ & & & & & & -\lambda & & \\ & & & & & & & \ddots & \\ & & & & & & & & -\lambda \end{pmatrix}$$

(4)
$$\begin{pmatrix} \lambda & \varepsilon & & & & & \\ 0 & \lambda & & & & & \\ & & \ddots & & & & \\ & & & \lambda & & & \\ & & & & -\lambda & & \\ & & & & & \ddots & \\ & & & & & & -\lambda \end{pmatrix} \qquad \varepsilon = 1 \text{ or } -1$$

In the expression above, if A has only one real eigenvalue λ, we delete $-\lambda$.

Proof. The case of (2.11)-(1) has already been studied in theorem 2.3. We consider the case of (2.11)-(2). In this case A has two complex eigenvalues $\mu = a + \sqrt{-1}b$ and $\bar{\mu}$ with multiplicity 1 and at least one real eigenvalue. Let $\lambda_1, \ldots, \lambda_r$ be the distinct real eigenvalues of A with algebraic multiplicities $m_1, \ldots, m_r$, respectively. Then the dimension of the eigenspace of λ_i coincides with m_i for each $i \in \{1, \ldots, r\}$. Applying the identity (2.6), we have

$$\begin{aligned} 0 &= \sum_{j \neq i} m_j \frac{\lambda_j + \lambda_i}{\lambda_j - \lambda_i} + \frac{\mu + \lambda_i}{\mu - \lambda_i} + \frac{\bar{\mu} + \lambda_i}{\bar{\mu} - \lambda_i} \\ &= \sum_{j \neq i} m_j \frac{\lambda_j^2 - \lambda_i^2}{(\lambda_j - \lambda_i)^2} + \frac{2\left(|\mu|^2 - \lambda_i^2\right)}{(\mu - \lambda_i)(\bar{\mu} - \lambda_i)} \qquad \text{for each } i. \end{aligned} \tag{2.12}$$

This implies that $|\mu|^2 = \lambda_1^2 = \cdots = \lambda_r^2$ and in particular $r = 1$ or 2. In fact we assume that there exists a $j \in \{1, \ldots, r\}$ such that $|\mu|^2 > \lambda_j^2$ (resp. $|\mu|^2 < \lambda_j^2$). Then we choose $i \in \{1, \ldots, r\}$ such that λ_i^2 is the minimum (resp. the maximum) of $\left\{\lambda_1^2, \ldots, \lambda_r^2\right\}$. For this i, the right-hand side of (2.12) is positive (resp. negative) and it is a contradiction. Consequently we obtain the form (2) in our theorem. We can prove the other cases by the similar argument. □

We devote the rest of this section to the preparation of the proof of theorem 1.1. The Ricci operator Q and their higher covariant derivatives $\nabla^i Q, i = 1, 2, \ldots$ are essential local invariants of a conformally flat semi-Riemannian manifold. We denote by $\mathfrak{so}(T_pM)$ the Lie algebra of the orthogonal group $O(T_pM)$ consisting of orthogonal transformations on T_pM. For a non-negative integer ℓ, we define a Lie subalgebra $\mathfrak{g}_\ell(p)$ of $\mathfrak{so}(T_pM)$ by

$$\mathfrak{g}_\ell(p) = \{A \in \mathfrak{so}(T_pM) | A \cdot Q_p = 0, A \cdot \nabla Q_p = 0, \ldots, A \cdot \nabla^\ell Q_p = 0\}, \tag{2.13}$$

where A acts as a derivation on the tensor algebra on T_pM. In particular,

$$\mathfrak{g}_0(p) = \{A \in \mathfrak{so}(T_pM) | A \cdot Q_p = 0\}. \tag{2.14}$$

If M is a homogeneous semi-Riemannian manifold, $\mathfrak{g}_\ell(p)$ is isomorphic to $\mathfrak{g}_\ell(q)$ for every $p, q \in M$ and every non-negative integer ℓ. So we simply write $\mathfrak{g}_\ell$. By straightforward computation, we have the following.

Lemma 2.5. *Let M_1^3 be a three-dimensional Lorentzian manifold. If the Ricci operator Q_p at $p \in M$ has the form of case 2, case 3 or case 4 with $\lambda \neq \sigma$ in (1.1), then $\mathfrak{g}_0(p) = \{0\}$.*

We state formulae for later use. On a three-dimensional conformally flat homogeneous Lorentzian manifold M, the following holds by (2.1) and (2.2):

$$R(X, Y)Z = (QX \wedge Y + X \wedge QY)Z - \frac{S}{2}(X \wedge Y)Z, \tag{2.15}$$

$$(\nabla_X Q)Y = (\nabla_Y Q)X. \tag{2.16}$$

3. The Ricci operator of the form case 2 in (1.1)

In this section, we study a three-dimensional simply connected conformally flat homogeneous Lorentzian manifold M_1^3 whose Ricci operator Q has the form

$$Q = \begin{pmatrix} a & -b & \\ b & a & \\ & & \lambda \end{pmatrix} b \neq 0, \tag{3.1}$$

with respect to a semi-orthonormal basis $\{e_1, e_2, e_3\}$, $\langle e_1, e_2\rangle = 1$, $\langle e_3, e_3\rangle = 1$. Then by lemma 2.5, $\mathfrak{g}_0 = \{0\}$. Therefore M is a three-dimensional Lie group with a left invariant Lorentzian metric. Let $\{e_1, e_2, e_3\}$ be left invariant semi-orthonormal frame fields on M with respect to which the Ricci operator has the form (3.1). We denote by $\{\Gamma_{ij}^k\}(i, j, k = 1, 2, 3)$ the connection functions, i.e., $\nabla_{e_i} e_j = \sum_{k=1}^3 \Gamma_{ij}^k e_k$. We note that Γ_{ij}^k are constant on M. Simply we denote by Γ_k the matrix whose (i, j)-components are Γ_{kj}^i. Since $\{e_1, e_2, e_3\}$ are semi-orthonormal frame fields, we have

$$\Gamma_k = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & -a_{11} & -a_{31} \\ a_{31} & -a_{13} & 0 \end{pmatrix}.$$

Using equation (2.16), we shall determine Γ_{ij}^k. Calculating $\nabla_{e_i} Q = [\Gamma_i, Q]$, we obtain

$$\nabla_{e_i} Q = \begin{pmatrix} 0 & -2b\Gamma_{i1}^1 & (\lambda - a)\Gamma_{i3}^1 - b\Gamma_{i1}^3 \\ -2b\Gamma_{i1}^1 & 0 & -b\Gamma_{i3}^1 - (\lambda - a)\Gamma_{i1}^3 \\ -b\Gamma_{i3}^1 - (\lambda - a)\Gamma_{i1}^3 & (\lambda - a)\Gamma_{i3}^1 - b\Gamma_{i1}^3 & 0 \end{pmatrix}.$$

Since $(\nabla_{e_i} Q)e_j = (\nabla_{e_j} Q)e_i$, we have

$$(i, j) = (1, 2) \qquad \begin{pmatrix} -2b\Gamma_{11}^1 \\ 0 \\ (\lambda - a)\Gamma_{13}^1 - b\Gamma_{11}^3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2b\Gamma_{21}^1 \\ -b\Gamma_{23}^1 - (\lambda - a)\Gamma_{21}^3 \end{pmatrix},$$

$$(i, j) = (1, 3) \qquad \begin{pmatrix} (\lambda - a)\Gamma_{13}^1 - b\Gamma_{11}^3 \\ -b\Gamma_{13}^1 - (\lambda - a)\Gamma_{11}^3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -2b\Gamma_{31}^1 \\ -b\Gamma_{33}^1 - (\lambda - a)\Gamma_{31}^3 \end{pmatrix},$$

$$(i, j) = (2, 3) \qquad \begin{pmatrix} (\lambda - a)\Gamma_{23}^1 - b\Gamma_{21}^3 \\ -b\Gamma_{23}^1 - (\lambda - a)\Gamma_{21}^3 \\ 0 \end{pmatrix} = \begin{pmatrix} -2b\Gamma_{31}^1 \\ 0 \\ (\lambda - a)\Gamma_{33}^1 - b\Gamma_{31}^3 \end{pmatrix}.$$

From these, we have

$$\Gamma_1 = \begin{pmatrix} 0 & 0 & \alpha c \\ 0 & 0 & -\beta c \\ \beta c & -\alpha c & 0 \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 0 & 0 & -\beta c \\ 0 & 0 & -\alpha c \\ \alpha c & \beta c & 0 \end{pmatrix}, \qquad \Gamma_3 = \begin{pmatrix} c & 0 & 0 \\ 0 & -c & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\alpha = 2b^2/((\lambda - a)^2 + b^2)$, $\beta = 2b(\lambda - a)/((\lambda - a)^2 + b^2)$ and c is a constant. From this, we have

$$\begin{array}{lll} \nabla_{e_1} e_1 = \beta c e_3 & \nabla_{e_2} e_1 = \alpha c e_3 & \nabla_{e_3} e_1 = c e_1 \\ \nabla_{e_1} e_2 = -\alpha c e_3 & \nabla_{e_2} e_2 = \beta c e_3 & \nabla_{e_3} e_2 = -c e_2 \\ \nabla_{e_1} e_3 = \alpha c e_1 - \beta c e_2 & \nabla_{e_2} e_3 = -\beta c e_1 - \alpha c e_2 & \nabla_{e_3} e_3 = 0 \end{array}$$

$$\begin{aligned} [e_1, e_2] &= -2\alpha c e_3 \\ [e_2, e_3] &= -\beta c e_1 + (c - \alpha c) e_2 \\ [e_3, e_1] &= (c - \alpha c) e_1 + \beta c e_2. \end{aligned}$$

We calculate the curvature tensors by two ways. First using the connection, we have

$$\begin{array}{ll} R(e_1, e_2)e_1 = 4\alpha c^2 e_1 & R(e_1, e_3)e_1 = 2\beta c^2 e_3 \\ R(e_1, e_2)e_2 = -4\alpha c^2 e_2 & R(e_1, e_3)e_2 = 2\alpha c^2 e_3 \\ R(e_1, e_2)e_3 = 0 & R(e_1, e_3)e_3 = -2\alpha c^2 e_1 - 2\beta c^2 e_2 \\ R(e_2, e_3)e_1 = 2\alpha c^2 e_3 & \\ R(e_2, e_3)e_2 = -2\beta c^2 e_3 & \\ R(e_2, e_3)e_3 = 2\beta c^2 e_1 - 2\alpha c^2 e_2. & \end{array} \tag{3.2}$$

Here we use $\alpha^2 + \beta^2 = 2\alpha$. On the other hand, we calculate the curvature tensors using equation (2.15), and have

$$\begin{array}{lll} R(e_1, e_2)e_1 = \left(a - \frac{\lambda}{2}\right) e_1 & R(e_1, e_3)e_1 = -b e_3 & R(e_2, e_3)e_1 = -\frac{\lambda}{2} e_3 \\ R(e_1, e_2)e_2 = -\left(a - \frac{\lambda}{2}\right) e_2 & R(e_1, e_3)e_2 = -\frac{\lambda}{2} e_3 & R(e_2, e_3)e_2 = b e_3 \\ R(e_1, e_2)e_3 = 0 & R(e_1, e_3)e_3 = \frac{\lambda}{2} e_1 + b e_2 & R(e_2, e_3)e_3 = -b e_1 + \frac{\lambda}{2} e_2. \end{array} \tag{3.3}$$

Comparing (3.2) and (3.3), we obtain $a = -\lambda/2, b = \pm(\sqrt{3}/2)\lambda, \lambda < 0$. In fact, $\alpha = 1/2$, $\beta = \pm\sqrt{3}/2, a = c^2, b = \mp\sqrt{3}c^2, \lambda = -2c^2$ and $c \neq 0$. When $\beta = \sqrt{3}/2$, for the semi-orthonormal basis $\{e_1, e_2, e_3\}$ the brackets $[e_i, e_j]$ become

$$\begin{aligned} [e_1, e_2] &= -c e_3 \\ [e_2, e_3] &= -\frac{\sqrt{3}}{2} c e_1 + \frac{c}{2} e_2 \\ [e_3, e_1] &= \frac{c}{2} e_1 + \frac{\sqrt{3}}{2} c e_2. \end{aligned} \tag{3.4}$$

This Lie algebra is semi-simple and isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Hence the Lie group is isomorphic to the universal covering $\widetilde{SL}(2, \mathbb{R})$ of $SL(2, \mathbb{R})$. The Ricci operator is of the form

$$\begin{pmatrix} c^2 & \sqrt{3}c^2 & 0 \\ -\sqrt{3}c^2 & c^2 & 0 \\ 0 & 0 & -2c^2 \end{pmatrix} c \neq 0$$

with respect to the semi-orthonormal basis $\{e_1, e_2, e_3\}$. When $\beta = -\sqrt{3}/2$, we may take the basis $\{e_2, e_1, -e_3\}$.

Conversely, for the basis $\{e_1, e_2, e_3\}$ of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ with the bracket operation [,] given by (3.4), we define a Lorentzian inner product by $\langle e_1, e_2\rangle = 1$, $\langle e_3, e_3\rangle = 1$, the others $= 0$. Then we go backward on the way of our calculation and can show that the Lie group with the left invariant Lorentzian metric above is conformally flat. Thus we have the following.

Proposition 3.1. *A three-dimensional simply connected conformally flat homogeneous Lorentzian manifold whose Ricci operator has the form (3.1) is isometric to the Lorentzian manifold of (4) in theorem 1.1.*

4. The Ricci operator of the form case 3 in (1.1)

In this section, we study a three-dimensional simply connected conformally flat homogeneous Lorentzian manifold M_1^3 whose Ricci operator Q has the form

$$Q = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 1 & 0 & \lambda \end{pmatrix} \tag{4.1}$$

with respect to a semi-orthonormal basis $\{e_1, e_2, e_3\}$, $\langle e_1, e_2\rangle = 1$, $\langle e_3, e_3\rangle = 1$. Then by lemma 2.5, $\mathfrak{g}_0 = \{0\}$. Therefore it is a three-dimensional Lie group with a left invariant Lorentzian metric. Let $\{e_1, e_2, e_3\}$ be left invariant semi-orthonormal frame fields with respect to which the Ricci operator has the form (4.1). We trace the same way as section 3 to determine the connection functions $\{\Gamma_{ij}^k\}$. Using equation (2.16), we have

$$\Gamma_1 = \begin{pmatrix} 2a & 0 & -2b \\ 0 & -2a & -c \\ c & 2b & 0 \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -b \\ b & 0 & 0 \end{pmatrix}, \qquad \Gamma_3 = \begin{pmatrix} -4b & 0 & 0 \\ 0 & 4b & -a \\ a & 0 & 0 \end{pmatrix},$$

where a, b and c are some constants. We calculate the curvature tensors using the connection functions and have

$$\langle R(e_1, e_2)e_3, e_1\rangle = ab \qquad \text{and} \qquad \langle R(e_1, e_3)e_2, e_1\rangle = 10ab.$$

On the other hand, we calculate them using equation (2.15) and have

$$\langle R(e_1, e_2)e_3, e_1\rangle = -1 \qquad \text{and} \qquad \langle R(e_1, e_3)e_2, e_1\rangle = -1.$$

From these, it follows that $ab = -1$ and $10ab = -1$, which is a contradiction. Thus we obtain the following.

Proposition 4.1. *There is no three-dimensional simply connected conformally flat homogeneous Lorentzian manifold whose Ricci operator has the form (4.1).*

5. The Ricci operator of the form case 4 in (1.1)

In this section, we study a three-dimensional simply connected conformally flat homogeneous Lorentzian manifold M_1^3 whose Ricci operator Q has the form

$$Q = \begin{pmatrix} \lambda & \varepsilon & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \sigma \end{pmatrix} \qquad \varepsilon = 1 \text{ or } -1 \tag{5.1}$$

with respect to a semi-orthonormal basis $\{e_1, e_2, e_3\}$. We consider the following two subcases:

case 4-1. $\lambda \neq \sigma$,
case 4-2. $\lambda = \sigma$.

Case 4-1. We will prove that this case does not occur by the same way as the previous section. By lemma 2.5, $\mathfrak{g}_0 = \{0\}$ and hence it is a three-dimensional Lie group with a left invariant Lorentzian metric. Let $\{e_1, e_2, e_3\}$ be left invariant semi-orthonormal frame fields with respect to which the Ricci operator has the form (5.1). We solve the connection functions $\{\Gamma_{ij}^k\}$ which satisfy equation (2.16) and have

$$\Gamma_1 = O, \qquad \Gamma_2 = \begin{pmatrix} a & 0 & b \\ 0 & -a & 0 \\ 0 & -b & 0 \end{pmatrix}, \qquad \Gamma_3 = \begin{pmatrix} \alpha b & 0 & 0 \\ 0 & -\alpha b & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\alpha = (\sigma - \lambda)/(2\varepsilon)$ and a, b are some constants. We calculate the curvature tensors using the connection functions and have

$$R(e_1, e_2)e_1 = 0 \quad \text{and} \quad R(e_1, e_3)e_2 = 0.$$

On the other hand, by (2.15), we have

$$R(e_1, e_2)e_1 = \left(\lambda - \frac{\sigma}{2}\right) e_2 \quad \text{and} \quad R(e_1, e_3)e_2 = -\frac{\sigma}{2} e_3.$$

From these, we have $\lambda = \sigma = 0$. This contradicts our assumption $\lambda \neq \sigma$. Thus we obtain the following.

Proposition 5.1. *There is no three-dimensional simply connected conformally flat homogeneous Lorentzian manifold whose Ricci operator has the form (5.1) with* $\lambda \neq \sigma$.

Case 4-2. In this case,

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} 0 & 0 & -s \\ 0 & 0 & 0 \\ 0 & s & 0 \end{pmatrix} \middle| \, s \in \mathbb{R} \right\}. \tag{5.2}$$

Let $\{e_1, e_2, e_3\}$ be local semi-orthonormal frame fields with respect to which the Ricci operator Q has the form

$$\begin{pmatrix} \lambda & \varepsilon & \\ & \lambda & \\ & & \lambda \end{pmatrix} \qquad \varepsilon = 1 \text{ or } -1. \tag{5.3}$$

We calculate the covariant derivative ∇Q of the Ricci operator and obtain

$$\nabla_{e_i} Q = \begin{pmatrix} 0 & 2\varepsilon\Gamma_{i1}^1 & \varepsilon\Gamma_{i1}^3 \\ 0 & 0 & 0 \\ 0 & \varepsilon\Gamma_{i1}^3 & 0 \end{pmatrix}. \tag{5.4}$$

By (2.16), we have

$$\Gamma_1 = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & -a & 0 \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} b & 0 & c \\ 0 & -b & -2d \\ 2d & -c & 0 \end{pmatrix}, \qquad \Gamma_3 = \begin{pmatrix} d & 0 & e \\ 0 & -d & 0 \\ 0 & -e & 0 \end{pmatrix}, \tag{5.5}$$

where a, b, c, d and e are not constant in general. We define the subbundle T_0 and T_{00} of the tangent bundle TM by $T_0 = \operatorname{Ker}(Q - \lambda \mathrm{Id})$ and $T_{00} = \operatorname{Im}(Q - \lambda\, \mathrm{Id})$, respectively. Then e_1

and e_3 are local frame fields of T_0 and e_1 is a local frame field of T_{00}. For the subbundle T_{00}, we define a $\mathrm{Hom}(T_{00}, TM/T_{00})$-valued 1-form α by

$$\alpha(X)(\xi) = \pi(\nabla_X \xi) \qquad \text{for} \quad X \in \Gamma(TM) \quad \text{and} \quad \xi \in \Gamma(T_{00}),$$

where π denotes the projection of TM onto TM/T_{00} and $\Gamma(TM)$ and $\Gamma(T_{00})$ denote the spaces of smooth sections of the vector bundles TM and T_{00}, respectively. The bundle T_{00} is parallel with respect to the Levi-Civita connection ∇ if and only if α vanishes. The bundles T_0 and T_{00} are invariant by the action of isometries. Since M is a homogeneous Lorentzian manifold, if $\alpha_p = 0$ at some point $p \in M$, then $\alpha = 0$ everywhere on M. We divide the case 4-2 into the following subcases:

case 4-2-(i). T_{00} is not parallel,
case 4-2-(ii). T_{00} is parallel.

By (5.5), we have

$$\alpha(e_1)(e_1) = 0, \qquad \alpha(e_2)(e_1) = 2d\pi(e_3), \qquad \alpha(e_3)(e_1) = 0.$$

Therefore $\alpha = 0$ if and only if $d = 0$.

Case 4-2-(i). First we consider the case when T_{00} is not parallel. Then $d \neq 0$. In this case, if $X \cdot \nabla Q = 0$ for $X \in \mathfrak{g}_0$, that is,

$$X\big((\nabla_{e_i} Q)e_j\big) - \big(\nabla_{Xe_i} Q\big)e_j - \big(\nabla_{e_i} Q\big)Xe_j = 0 \qquad i, j = 1, 2, 3,$$

then we have $X = 0$. Therefore we have $\mathfrak{g}_1 = \{0\}$ and hence M is a three-dimensional Lie group with a left invariant Lorentzian metric. Let $\{e_1, e_2, e_3\}$ be left invariant semi-orthonormal frame fields with respect to which the Ricci operator has the form (5.3). Then the connection functions Γ_{ij}^k are constant and in particular, a, b, c, d and e in (5.5) are constant.

Lemma 5.2. *We can choose left invariant semi-orthonormal frame fields* $\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3\}$ *with* $\Gamma_{21}^1 = 0$.

Proof of lemma 5.2. The one-dimensional Lie subgroup of $SO(1,2)$ corresponding to $\mathfrak{g}_0((5.2))$ is given by

$$\left\{ \begin{pmatrix} 1 & -\frac{1}{2}s^2 & -s \\ 0 & 1 & 0 \\ 0 & s & 1 \end{pmatrix} \middle| \, s \in \mathbb{R} \right\}.$$

We define new semi-orthonormal frame fields $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ by

$$\tilde{e}_1 = e_1, \qquad \tilde{e}_2 = -\frac{1}{2}s^2 e_1 + e_2 + s e_3, \qquad \tilde{e}_3 = -s e_1 + e_3.$$

Then the Ricci operator has the form (5.3) with respect to $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$. Moreover we have

$$\nabla_{\tilde{e}_2} \tilde{e}_1 = (b + 3sd)\tilde{e}_1 + 2d\tilde{e}_3.$$

Therefore if we put $s = -b/(3d)$, we obtain $\tilde{\Gamma}_{21}^1 = 0$. □

From now on, we use left invariant semi-orthonormal frame fields $\{e_1, e_2, e_3\}$ defined by lemma 5.2. Then we have

$$\begin{array}{lll} \nabla_{e_1} e_1 = 0 & \nabla_{e_2} e_1 = 2de_3 & \nabla_{e_3} e_1 = de_1 \\ \nabla_{e_1} e_2 = -ae_3 & \nabla_{e_2} e_2 = -ce_3 & \nabla_{e_3} e_2 = -de_2 - ee_3 \\ \nabla_{e_1} e_3 = ae_1 & \nabla_{e_2} e_3 = ce_1 - 2de_2 & \nabla_{e_3} e_3 = ee_1 \end{array}$$

$$[e_1, e_2] = -(a + 2d)e_3$$

$$[e_2, e_3] = ce_1 - de_2 + ee_3$$

$$[e_3, e_1] = -(a - d)e_1,$$

where a, c, d, e are constant and $d \neq 0$. Calculating the curvature tensors using the connection, we have

$$
\begin{aligned}
&R(e_1,e_2)e_1 = (3a+2d)de_1 && R(e_1,e_3)e_1 = 0\\
&R(e_1,e_2)e_2 = -(3a+2d)de_2 - (a+2d)ee_3 && R(e_1,e_3)e_2 = a^2e_3\\
&R(e_1,e_2)e_3 = (a+2d)ee_1 && R(e_1,e_3)e_3 = -a^2e_1\\
&R(e_2,e_3)e_1 = -3dee_1 + 4d^2e_3\\
&R(e_2,e_3)e_2 = 3dee_2 + (ac+e^2)e_3\\
&R(e_2,e_3)e_3 = -(ac+e^2)e_1 - 4d^2e_2.
\end{aligned}
\tag{5.6}
$$

On the other hand, we calculate the curvature tensors using equation (2.15) and have

$$
\begin{aligned}
&R(e_1,e_2)e_1 = \frac{\lambda}{2}e_1 && R(e_1,e_3)e_1 = 0 && R(e_2,e_3)e_1 = -\frac{\lambda}{2}e_3\\
&R(e_1,e_2)e_2 = -\frac{\lambda}{2}e_2 && R(e_1,e_3)e_2 = -\frac{\lambda}{2}e_3 && R(e_2,e_3)e_2 = -\varepsilon e_3\\
&R(e_1,e_2)e_3 = 0 && R(e_1,e_3)e_3 = \frac{\lambda}{2}e_1 && R(e_2,e_3)e_3 = \varepsilon e_1 + \frac{\lambda}{2}e_2.
\end{aligned}
\tag{5.7}
$$

Comparing (5.6) and (5.7), we obtain $a = -2d$, $c = \varepsilon/(2d)$, $e = 0$, and $\lambda = -8d^2 (d \neq 0)$. Hence the brackets $[e_i, e_j]$ are given by

$$
\begin{aligned}
&[e_1,e_2] = 0\\
&[e_2,e_3] = \frac{\varepsilon}{2d}e_1 - de_2 \qquad \varepsilon = 1 \text{ or } -1\\
&[e_3,e_1] = 3de_1.
\end{aligned}
\tag{5.8}
$$

This Lie algebra is nonunimodular and its unimodular kernel is spanned by e_1 and e_2. Moreover the Ricci operator has the form

$$
\begin{pmatrix} -8d^2 & \varepsilon & \\ & -8d^2 & \\ & & -8d^2 \end{pmatrix} \qquad d \neq 0.
$$

Conversely, for the basis $\{e_1, e_2, e_3\}$ of the Lie algebra with the bracket operation given by (5.8), we define a Lorentzian inner product by $\langle e_1, e_2\rangle = 1$, $\langle e_3, e_3\rangle = 1$, the others $= 0$. Then we go backward on the way of our calculation and can show that the Lie group with the left invariant Lorentzian metric is conformally flat.

Case 4-2-(ii). We consider the case when the bundle T_{00} is parallel. Then $d = 0$ in (5.5). In this case,

$$
\mathfrak{g}_0 = \mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & 0 & -s\\ 0 & 0 & 0\\ 0 & s & 0 \end{pmatrix} \middle|\, s \in \mathbb{R} \right\}.
$$

Moreover, the following holds.

Lemma 5.3. *The subbundle* $T_0 = \mathrm{Ker}(Q - \lambda\,\mathrm{Id})$ *is also parallel and* $\lambda = 0$.

Proof of lemma 5.3. We note that T_0 is the subbundle of TM which is the orthogonal complement of T_{00}. Since T_{00} is parallel, then T_0 is also parallel. Let $\{e_1, e_2, e_3\}$ be a semi-orthonormal basis with respect to which the Ricci operator has the form (5.3). Since T_{00} is parallel, $R(e_2, e_3)e_1 = \alpha e_1$ for some $\alpha \in \mathbb{R}$. On the other hand by (2.15), $R(e_2, e_3)e_1 = -(\lambda/2)e_3$. Therefore we have $\lambda = 0$. □

By this lemma and the previous arguments in this section, we obtain the following characterization.

Proposition 5.4. *A three-dimensional simply connected conformally flat homogeneous Lorentzian manifold whose Ricci operator has the form (5.3) with $\lambda \neq 0$ is isometric to the Lorentzian manifold of (5) in theorem 1.1.*

We will show the global description of case 4-2-(ii) in the next section.

6. Global description of case 4 with $\lambda = \sigma = 0$ in (1.1)

In this section we study a three-dimensional simply connected conformally flat homogeneous Lorentzian manifold M_1^3 whose Ricci operator Q has the form

$$Q = \begin{pmatrix} 0 & \varepsilon & \\ & 0 & \\ & & 0 \end{pmatrix} \qquad \varepsilon = 1 \text{ or } -1 \tag{6.1}$$

with respect to a semi-orthonormal basis $\{e_1, e_2, e_3\}$. In this section, our main result is the following:

Theorem 6.1. *Let M_1^3 be a Lorentzian manifold which satisfies the assumption above. Then M_1^3 is isometric to the Lorentzian manifold of (6) in theorem 1.1.*

At first we will show the outline of the proof of theorem 6.1. As we refered in the introduction of this paper, in the previous paper [4], we have shown how to construct the examples which are conformally flat semi-Riemannian manifolds whose Ricci operators Q satisfy $Q^2 = 0$. Moreover we obtained the characterization of such examples ([4] theorem 5.4). We recall it. Let M be an n-dimensional conformally flat semi-Riemannian manifold with $Q^2 = 0$. We assume that the rank of the Ricci operator Q is k everywhere on M. Then the distribution $T_0 = \operatorname{Ker} Q$ is completely integrable and its leaves are totally geodesic with respect to the Levi-Civita connection ∇. Suppose that the distribution T_0 is parallel on M with respect to the Levi-Civita connection ∇ and that each leaf of T_0 is geodesically complete with respect to the induced connection. Then there exists a k-dimensional manifold N and a centro-affine hypersurface immersion $F : N \to \mathbb{R}^{k+1} - \{0\}$ such that M is isometric to the semi-Riemannian manifold constructed from (N, F) by the method in the introduction.

Now we return to the proof of theorem 6.1. We have already shown that the distribution $T_0 = \operatorname{Ker} Q$ is parallel in lemma 5.3. Therefore to prove theorem 6.1, it is sufficient that we show the following two:

1. to prove that each leaf of T_0 is geodesically complete with respect to the induced connection.
2. to classify homogeneous centro-affine plane curves in $\mathbb{R}^2$.

We will study the first problem above. Let M_1^3 be a Lorentzian manifold which satisfies the assumption in the beginning of this section. Let K be a connected Lie group which acts isometrically, transitively, and effectively on M. For $k \in K$, we denote by τ_k the diffeomorphism of M defined by k. Let $\mathfrak{k}$ be the Lie algebra of K and for $X \in \mathfrak{k}$ we denote by X^* the vector field on M generated by X, i.e., the vector field defined by the one-parameter group $\{\tau_{\exp tX} | t \in \mathbb{R}\}$ of transformations of M. We fix a point of M which is called the origin of M and denoted by o. The isotropic subgroup of K at the origin o is denoted by H and the Lie subalgebra corresponding to H is denoted by $\mathfrak{h}$. Then the quotient manifold K/H is diffeomorphic to M. We denote by $\pi : K \to M$ the natural projection defined by $\pi(k) = \tau_k(o)$ for $k \in K$ and by the same symbol $\pi : \mathfrak{k} \to T_o M$ its differentiation π_{*e} at $e \in K$. Then we have $\pi(X) = X_o^*$. We denote by $\lambda : H \to \operatorname{End}(T_o M)$ the linear isotropy representation defined by $\lambda(h) = \tau_{h*o}$ and by the same symbol $\lambda : \mathfrak{h} \to \mathrm{End}(T_o M)$ its corresponding Lie algebra homomorphism. Then we have

$$\begin{aligned} &\lambda(h)\pi(X) = \pi(\mathrm{Ad}(h)X) \\ &\lambda(A)\pi(X) = \pi([A, X]) \qquad \text{for} \quad h \in H, A \in \mathfrak{h} \quad \text{and} \quad X \in \mathfrak{k}. \end{aligned} \tag{6.2}$$

Under the assumption of this section, $\dim \mathfrak{g}_0 = 1$, where $\mathfrak{g}_0 = \{A \in \mathfrak{so}(T_o M) | A \cdot Q_o = 0\}$. Therefore we have $\dim \mathfrak{h} \leqslant 1$ and hence $\dim \mathfrak{k} = 4$ or 3. We study the first problem dividing into the two cases.

At first we assume that $\dim \mathfrak{k} = 4$. We will determine the structure of the Lie algebra $\mathfrak{k}$ and the connection Γ applying the theory of invariant affine connections. We refer to Kobayashi and Nomizu [5] chapter X. A semi-orthonormal basis $\{e_1, e_2, e_3\}$, $\langle e_1, e_2\rangle = 1$, $\langle e_3, e_3\rangle = 1$ of the tangent space $T_p M$, $p \in M$ is called *adapted* if $e_1 \in T_{00}(p) = \mathrm{Im} Q_p$ and $\{e_1, e_3\}_{\mathbb{R}} = T_0(p) = \mathrm{Ker}\, Q_p$. We denote by P the bundle of adapted semi-orthonormal bases over M. The structure group G of P and its Lie algebra $\mathfrak{g}$ are given as follows:

$$G = \left\{ \begin{pmatrix} a & -\dfrac{b^2}{2a} & b \\ 0 & \dfrac{1}{a} & 0 \\ 0 & -\dfrac{b}{a}\varepsilon & \varepsilon \end{pmatrix} \middle| \, a \neq 0, b \in \mathbb{R}, \varepsilon = 1 \text{ or } -1 \right\},$$

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & -a & 0 \\ 0 & -b & 0 \end{pmatrix} \middle| \, a, b \in \mathbb{R} \right\}.$$

Since T_0 and T_{00} are parallel with respect to the Levi-Civita connection, the Levi-Civita connection is reduced to a connection in the bundle P. We define a basis $\{E_1, E_2\}$ of the Lie algebra $\mathfrak{g}$ by

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}. \tag{6.3}$$

Then we have $[E_1, E_2] = E_2$. Let $u_o = \{e_1, e_2, e_3\}$ be a semi-orthonormal basis of $T_o M$ with respect to which the Ricci operator has the form (6.1). Then u_o is adapted. From now on, we fix this basis u_o and by u_o we identify $\mathbb{R}^3$ with $T_o M$. Then $\mathfrak{g}_0 = \{A \in \mathfrak{so}(T_o M) | A \cdot Q_o = 0\}$ is spanned by E_2 and for the linear isotropy representation λ of $\mathfrak{h}$, we have $\lambda(\mathfrak{h}) = \mathfrak{g}_0$.

We recall the theory of invariant affine connections (see section 1 in [5] chapter X). For the Levi-Civita connection reduced to P, there exists a linear map $\Gamma : \mathfrak{k} \to \mathfrak{g}$ such that the following equations hold:

$$\Gamma(A) = \lambda(A) \qquad \text{for} \quad A \in \mathfrak{h}, \tag{6.4}$$

$$\Gamma([A, X]) = [\lambda(A), \Gamma(X)] \qquad \text{for} \quad A \in \mathfrak{h}, X \in \mathfrak{k}, \tag{6.5}$$

$$\Gamma(X)\pi(Y) - \Gamma(Y)\pi(X) = \pi([X, Y]) \qquad \text{for} \quad X, Y \in \mathfrak{k}, \tag{6.6}$$

$$R(\pi(X), \pi(Y)) = [\Gamma(X), \Gamma(Y)] - \Gamma([X, Y]) \qquad \text{for} \quad X, Y \in \mathfrak{k}, \tag{6.7}$$

where we denote by R the curvature tensor at $o \in M$ and note that we identify $\mathbb{R}^3$ with $T_o M$ by the basis u_o. Moreover for $X, Y \in \mathfrak{k}$ and $k \in K$, the following holds:

$$(\nabla_{X^*} Y^*)_{\tau_k(o)} = \tau_{k_*}(\Gamma(\mathrm{Ad}(k^{-1})Y)\pi(\mathrm{Ad}(k^{-1})X)). \tag{6.8}$$

For the basis $\{E_1, E_2\}$ of $\mathfrak{g}$, we put $\Gamma(X) = \Gamma_1(X)E_1 + \Gamma_2(X)E_2$ for $X \in \mathfrak{k}$. We choose an element $A \in \mathfrak{h}$ such that $\lambda(A) = E_2$. For the basis $u_o = \{e_1, e_2, e_3\}$, we choose $X_1, X_2, X_3 \in \mathfrak{k}$ such that $\pi(X_i) = e_i$ and $\Gamma_2(X_i) = 0\,(i = 1, 2, 3)$. By (6.2), (6.5), (6.6) and (6.7), we will determine the bracket operation [,] of $\mathfrak{k}$ and the connection $\Gamma : \mathfrak{k} \to \mathfrak{g}$.

Lemma 6.2. *For the basis $\{A, X_1, X_2, X_3\}$ of $\mathfrak{k}$ given above, we have*

$$[A, X_1] = 0, \qquad [A, X_2] = -cA - X_3, \qquad [A, X_3] = X_1$$
$$[X_1, X_2] = -cX_1, \qquad [X_2, X_3] = -\varepsilon A, \qquad [X_1, X_3] = 0$$
$$\Gamma(X_1) = 0, \qquad \Gamma(X_2) = cE_1, \qquad \Gamma(X_3) = 0,$$

where c is a constant and $\varepsilon = 1$ or -1.

Proof of lemma 6.2. By (6.2), there exist real numbers c_1, c_2 and c_3 such that $[A, X_1] = c_1A$, $[A, X_2] = c_2A - X_3$ and $[A, X_3] = c_3A + X_1$. By (6.5), we have $c_i = -\Gamma_1(X_i)(i = 1, 2, 3)$ and $\Gamma_1(X_1) = \Gamma_1(X_3) = 0$. So we put $c = \Gamma_1(X_2)$ newly and obtain

$$[A, X_1] = 0, \qquad [A, X_2] = -cA - X_3, \qquad [A, X_3] = X_1$$
$$\Gamma(X_1) = \Gamma(X_3) = 0, \qquad \Gamma(X_2) = cE_1.$$

By (6.6), there exist real numbers b_1, b_2 and b_3 such that

$$[X_1, X_2] = b_3A - cX_1, \qquad [X_2, X_3] = b_1A, \qquad [X_3, X_1] = b_2A.$$

By (6.7) we have

$$R(\pi(X_1), \pi(X_2)) = -b_3E_2, \qquad R(\pi(X_2), \pi(X_3)) = -b_1E_2,$$
$$R(\pi(X_1), \pi(X_3)) = b_2E_2.$$

On the other hand, by (2.15)

$$R(\pi(X_1), \pi(X_2)) = 0, \qquad R(\pi(X_2), \pi(X_3)) = \varepsilon E_2, \qquad R(\pi(X_1), \pi(X_3)) = 0.$$

From these, it follows that $b_2 = b_3 = 0$ and $b_1 = -\varepsilon$. □

Corollary 6.3. *When* $\dim \mathfrak{k} = 4$, *each leaf of the distribution* $T_0 = \operatorname{Ker} Q$ *is geodesically complete with respect to the induced connection.*

Proof of corollary 6.3. Since the distribution T_0 is invariant by the action of isometries, it is sufficient to prove that a geodesic of M through the origin and tangent to $T_0(o)$ is defined on the whole of $\mathbb{R}$. To prove this, for an arbitrary vector $X = aX_1 + bX_3 \in \mathfrak{k}(a, b \in \mathbb{R})$, we will show that $\tau_{\exp tX}(o)$ is a geodesic of M tangent to $\pi(X) = ae_1 + be_3$. By lemma 6.2, $\Gamma(X) = a\Gamma(X_1) + b\Gamma(X_3) = 0$. We note that $\operatorname{Ad}(\exp tX)X = X$. By (6.8), we have

$$\begin{aligned}(\nabla_{X^*}X^*)_{\tau_{\exp tX}(o)} &= \tau_{\exp tX*}(\Gamma(\operatorname{Ad}(\exp(-tX))X)\pi(\operatorname{Ad}(\exp(-tX))X)\\ &= \tau_{\exp tX*}(\Gamma(X)\pi(X)) = 0.\end{aligned}$$

Therefore the integral curve $\tau_{\exp tX}(o)$ of X^* through the origin is a geodesic. In particular, it is defined on the whole of $\mathbb{R}$. □

Next we treat the case of $\dim \mathfrak{k} = 3$. Then M is a three-dimensional Lie group with a left invariant Lorentzian metric. We will investigate it by the same way as section 5. Let $\{e_1, e_2, e_3\}$ be left invariant semi-orthonormal frame fields with respect to which the Ricci operator has the form (6.1) and $\left\{\Gamma_{ij}^k\right\}$ the connection functions. By the same calculations as section 5, we obtain the following.

Lemma 6.4. *We can choose left invariant semi-orthonormal frame fields* $\{e_1, e_2, e_3\}$ *such that*

$$[e_1, e_2] = -ae_1, \qquad [e_2, e_3] = be_3, \qquad [e_3, e_1] = 0,$$

$$\Gamma_1 = 0, \qquad \Gamma_2 = aE_1, \qquad \Gamma_3 = bE_2,$$

where a, b *are some constants which satisfy* $(a - b)b = \varepsilon$ *and* E_1 *and* E_2 *are the elements of* $\mathfrak{g}$ *defined by (6.3).*

We will show the Lie group corresponding to the Lie algebra given in lemma 6.4. We denote by (s, x, y) the coordinates of $\mathbb{R}^3$. On $\mathbb{R}^3$, we define a product as follows:

$$(s_1, x_1, y_1)(s_2, x_2, y_2) = (s_1 + s_2, e^{-as_2}x_1 + x_2, e^{-bs_2}y_1 + y_2).$$

Then $\mathbb{R}^3$ equipped with the product is a Lie group, which is denoted by $\widetilde{K}$. We define vector fields $e_i (i = 1, 2, 3)$ on $\mathbb{R}^3$ by

$$e_1 = \frac{\partial}{\partial x}, \qquad e_2 = \frac{\partial}{\partial s} - ax\frac{\partial}{\partial x} - by\frac{\partial}{\partial y}, \qquad e_3 = \frac{\partial}{\partial y}.$$

Then they are left invariant vector fields on $\widetilde{K}$ and satisfy

$$[e_1, e_2] = -ae_1, \qquad [e_2, e_3] = be_3, \qquad [e_3, e_1] = 0.$$

Therefore the Lie algebra of $\widetilde{K}$ is isomorphic to the Lie algebra $\mathfrak{k}$. We define a left invariant Lorentzian metric on $\widetilde{K}$ such that $\langle e_1, e_2\rangle = 1$, $\langle e_3, e_3\rangle = 1$, the others $= 0$. Then $\widetilde{K}$ equipped with the left invariant Lorentzian metric is isometric to M. The distribution $T_0 = \operatorname{Ker} Q$ is spanned by $e_1 = \partial/\partial x$ and $e_3 = \partial/\partial y$. Therefore a leaf of T_0 through (s_o, x_o, y_o) is given by $\{(s_o, x, y)|x, y \in \mathbb{R}\}$. Let $M_0(o)$ be the leaf of T_0 through the origin which is given by $\{(0, x, y)|x, y \in \mathbb{R}\}$. We will determine the geodesic $\gamma(t)$ of $M_0(o)$ such that $\gamma(0) = o$ (the origin) and $\dot\gamma(0) = (0, p, q)$. The geodesic $\gamma(t) = (0, x(t), y(t))$ satisfies the system of equations

$$\begin{cases} \dfrac{d^2x}{dt^2} + b\left(\dfrac{dy}{dt}\right)^2 = 0 \\ \dfrac{d^2y}{dt^2} = 0. \end{cases}$$

We can easily solve it and obtain $x(t) = pt - (1/2)bq^2t^2$, $y(t) = qt$. In particular it is defined on the whole of $\mathbb{R}$. Thus the following has been proved.

Corollary 6.5. *When* $\dim \mathfrak{k} = 3$, *each leaf of the distribution* $T_0 = \operatorname{Ker} Q$ *is geodesically complete with respect to the induced connection.*

By corollaries 6.3 and 6.5, we have solved the first problem in the proof of theorem 6.1.

Now we will study the second problem in the proof of theorem 6.1. We recall some relations between the affine differential geometry of centro-affine hypersurface immersions and the semi-Riemannian geometry of conformally flat semi-Riemannian manifolds constructed from such hypersurface immersions (mainly section 3 in [4]). Let $F_i : N_i \to \mathbb{R}^{k+1} - \{0\} (i = 1, 2)$ be centro-affine hypersurface immersions of k-dimensional manifolds N_i and M_i be n-dimensional conformally flat semi-Riemannian manifolds constructed from (N_i, F_i), respectively, by the method recalled in the introduction of this paper. If (N_1, F_1) and (N_2, F_2) are $GL(k+1, \mathbb{R})$-congruent, that is, there exist a diffeomorphism a of N_1 onto N_2 and a linear transformation $\tilde a \in GL(k+1, \mathbb{R})$ such that $F_2 \circ a = \tilde a \circ F_1$, then M_1 is isometric to M_2 as semi-Riemannian manifolds. If the centro-affine fundamental form of (N, F) vanishes, then the semi-Riemannian manifold M constructed from (N, F) is flat. A centro-affine hypersurface immersion $F : N \to \mathbb{R}^{k+1} - \{0\}$ is called *homogeneous* if there exist a connected Lie group

K which acts transitively on N and a Lie group homomorphism $\rho : K \to GL(k+1, \mathbb{R})$ such that

$$F(ap) = \rho(a)F(p) \qquad \text{for} \quad a \in K, p \in N.$$

We consider a homogeneous centro-affine curve in $\mathbb{R}^2$. If it is non-degenerate, that is, it has a non-zero centro-affine fundamental form, the dimension of the corresponding Lie group K is equal to 1. Therefore our problem is reduced to the following:

to classify non-degenerate centro-affine curves which are orbits of points in $\mathbb{R}^2 - \{0\}$ under one-parameter subgroup of $GL(2, \mathbb{R})$ up to congruence by linear transformations in $GL(2, \mathbb{R})$.

It is easy to classify one-parameter subgroup of $GL(2, \mathbb{R})$ up to inner automorphisms and their orbits by linear algebra. Thus we obtain the following.

Proposition 6.6. *A homogeneous non-degenerate centro-affine curve in $\mathbb{R}^2$ is congruent to one of the following by linear transformations in $GL(2, \mathbb{R})$:*

(1) $y = x^{\lambda} (\lambda > 1, x > 0)$,

(2) $y = x^{\lambda} (\lambda \leqslant -1, x > 0)$,

(3) $\begin{cases} x = e^t \cos bt \\ y = e^t \sin bt \end{cases} \quad (b > 0)$,

(4) $x^2 + y^2 = 1$,

(5) $y = x \log x (x > 0)$.

This, combined with the previous arguments, proves our theorem 6.1.

7. The case of product manifolds

In this section we study a three-dimensional simply connected conformally flat homogeneous Lorentzian manifold M_1^3 whose Ricci operator Q has the form

$$(1) \quad \begin{pmatrix} k & & \\ & k & \\ & & 0 \end{pmatrix} \quad k \neq 0, \qquad (2) \quad \begin{pmatrix} 0 & & \\ & k & \\ & & k \end{pmatrix} \quad k \neq 0, \tag{7.1}$$

with respect to an orthonormal basis $\{e_1, e_2, e_3\}$, $\langle e_1, e_1 \rangle = -1$, $\langle e_2, e_2 \rangle = \langle e_3, e_3 \rangle = 1$. In section 2, we have shown that M_1^3 is locally a product manifold (theorem 2.3). In this section we show global properties.

Theorem 7.1. *Let M_1^3 be a Lorentzian manifold which satisfies the assumption above. If the Ricci operator has the form of (7.1)(1) (resp. (2)), then M_1^3 is isometric to $M_1^2(k) \times \mathbb{R}^1$ (resp. $\mathbb{R}_1^1 \times M^2(k)$).*

Proof. We will prove the case of (7.1)(1). The proof of the case (7.1)(2) is similar. We will trace the same way as the proof of theorem 6.1. Let K be a connected Lie group which acts isometrically, transitively, and effectively on M and $\mathfrak{k}$ be the Lie algebra of K. We fix a point of M, which is denoted by o. The isotropy subgroup of K at the origin o is denoted by H and the Lie algebra corresponding to H is denoted by $\mathfrak{h}$. Under the assumption of this section,

$$\mathfrak{g}_0 = \{A \in \mathfrak{so}(T_o M) | A \cdot Q_o = 0\} = \left\{ \begin{pmatrix} 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| \, a \in \mathbb{R} \right\}. \tag{7.2}$$

In particular $\dim \mathfrak{g}_0 = 1$. Therefore $\dim \mathfrak{h} \leqslant 1$ and hence $\dim \mathfrak{k} = 4$ or 3.

First we discuss the case of $\dim \mathfrak{k} = 4$. We define the subbundles of the tangent bundle TM by $T_k = \mathrm{Ker}(Q - k\,\mathrm{Id})$ and $T_0 = \mathrm{Ker}\, Q$. An orthonormal basis $\{e_1, e_2, e_3\}$ of the tangent space T_pM, $p \in M$ is called *adapted* if $\{e_1, e_2\}_{\mathbb{R}} = T_k(p)$ and $e_3 \in T_0(p)$. We denote by P the bundle of adapted orthonormal bases over M. Then the Lie algebra $\mathfrak{g}$ of the structure group G in P coincides with $\mathfrak{g}_0$ given by (7.2). As shown in the proof of theorem 2.3, T_k and T_0 are parallel with respect to the Levi-Civita connection ∇. Therefore the Levi-Civita connection is reduced to a connection in P. As in section 6, there exists a linear map $\Gamma : \mathfrak{k} \to \mathfrak{g}$ which satisfies (6.4)–(6.7). We define a subspace $\mathfrak{m}$ of $\mathfrak{k}$ by $\mathfrak{m} = \{X \in \mathfrak{k} | \Gamma(X) = 0\}$. Since $\lambda(\mathfrak{h}) = \mathfrak{g}$, we have a direct sum decomposition $\mathfrak{k} = \mathfrak{h} + \mathfrak{m}$. By the same argument as corollary 6.3, we see that the Lorentzian manifold M_1^3 is geodesically complete. Thus M_1^3 is a simply connected, complete Lorentzian manifold. By the decomposition theorem of de Rham and Wu ([8] appendix I), M_1^3 is isometric to the product manifold $M_1^2(k) \times \mathbb{R}^1$. □

Next we discuss the case of $\dim \mathfrak{k} = 3$. Then M_1^3 is a three-dimensional Lie group with a left invariant Lorentzian metric. Let $\{e_1, e_2, e_3\}$ be left invariant orthonormal frame fields with respect to which the Ricci operator has the form (7.1)(1) and $\left\{\Gamma_{ij}^k\right\}$ the connection functions.

Lemma 7.2. *We can choose left invariant orthonormal frame fields* $\{e_1, e_2, e_3\}$ *such that if* $k > 0$, $a = \sqrt{k}$,

$$[e_1, e_2] = ae_2, \qquad [e_2, e_3] = [e_3, e_1] = 0$$

$$\Gamma_1 = \Gamma_3 = 0, \qquad \Gamma_2 = -aE$$

and if $k < 0$, $a = \sqrt{-k}$,

$$[e_1, e_2] = ae_1, \qquad [e_2, e_3] = [e_3, e_1] = 0$$

$$\Gamma_1 = aE \qquad \Gamma_2 = \Gamma_3 = 0,$$

where

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof of lemma 7.2. Since T_k and T_0 are parallel with respect to ∇, there exist some constants a_1, a_2, a_3 such that $\Gamma_i = a_i E (i = 1, 2, 3)$. Then we have

$$[e_1, e_2] = a_1 e_1 - a_2 e_2, \qquad [e_2, e_3] = -a_3 e_1, \qquad [e_3, e_1] = a_3 e_2.$$

Calculating the curvature tensors using the connection, we have

$$R(e_1, e_2) = \left(-a_1^2 + a_2^2\right)E, \qquad R(e_2, e_3) = a_1 a_3 E, \qquad R(e_1, e_3) = a_2 a_3 E.$$

On the other hand, we calculate them using equation (2.15) and we have

$$R(e_1, e_2) = kE, \qquad R(e_2, e_3) = R(e_1, e_3) = 0.$$

Therefore $-a_1^2 + a_2^2 = k(\neq 0)$ and $a_1 a_3 = a_2 a_3 = 0$. Since at least one of a_1 and a_2 is not zero, $a_3 = 0$. We put $v = [e_1, e_2]$. Then $\langle v, v\rangle = k$. If $k > 0$, we put $e_2 = v/\sqrt{k}$ and define new orthonormal frame fields $\{e_1, e_2, e_3\}$. If $k < 0$, we put $e_1 = v/\sqrt{-k}$ and define new orthonormal frame fields $\{e_1, e_2, e_3\}$. Then they satisfy the conditions in lemma 7.2

We continue the proof of theorem 7.1 in the case of $\dim \mathfrak{k} = 3$. We put $\mathfrak{k}_0 = \{e_3\}_{\mathbb{R}}$ and $\mathfrak{k}_1 = \{e_1, e_2\}_{\mathbb{R}}$. Then by lemma 7.2, we have a Lie algebra direct sum $\mathfrak{k} = \mathfrak{k}_1 + \mathfrak{k}_0$. We denote by K_1 and K_0 the Lie subgroups of K which correspond to the Lie algebras $\mathfrak{k}_1$ and $\mathfrak{k}_0$, respectively. Then the Lie group K is isomorphic to the product Lie group $K_1 \times K_0$. Since the metric on $M_1^3 = K$ is left invariant, $M_1^3 = K$ is isometric to the product Lorentzian manifold $K_1 \times K_0$. Here K_1 is a two-dimensional simply connected homogeneous Lorentzian manifold of constant sectional curvature k. $\square$

Remark 7.3. The two-dimensional Lie group K_1 with left invariant Lorentzian metric in the case of $\dim \mathfrak{k} = 3$ is not geodesically complete. We can prove this fact straightforward but omit it.

References

[1] Chaichi M, García-Río E and Vázquez-Abal M E 2005 Three-dimensional Lorentz manifolds admitting a parallel null vector field *J. Phys. A: Math. Gen.* **38** 841–50

[2] Ferus D 1970 Totally geodesic foliations *Math. Ann.* **188** 313–6

[3] Hahn J 1984 Isoparametric hypersurfaces in the pseudo-Riemannian space forms *Math. Z.* **187** 195–208

[4] Honda K and Tsukada K 2004 Conformally flat semi-Riemannian manifolds with nilpotent Ricci operators and affine differential geometry *Ann. Glob. Anal. Geom.* **25** 253–75

[5] Kobayashi S and Nomizu K 1969 *Foundations of Differential Geometry* vol 2 (New York: Interscience)

[6] O'Neill B 1983 *Semi-Riemannian Geometry with Application to Relativity* (New York: Academic)

[7] Takagi H 1975 Conformally flat Riemannian manifolds admitting a transitive group of isometries: I, II *Tohoku Math. J.* **27** 103–110(I), 445–51(II)

[8] Wu H 1967 Holonomy groups of indefinite metrics *Pacific J. Math.* **20** 351–92